Stability of the inverses and Fredholm properties of interpolated operators

Mieczysław Mastyło

Adam Mickiewicz University, Poznań

XI Conference of the Georgian Mathematical Union
Batumi, August 23–28, 2021

Based on joint works with Irina Asekritova and Natan Kruglyak

Outline

Definition A bounded linear operator $T\colon X \to Y$ between Banach spaces is said to be semi-Fredholm if $T(X)$ is closed in Y and at least one of the subspaces $\ker T$, $Y/T(X)$ is finite-dimensional. Then the index of T is given by

$$\mathrm{ind}(T) := \dim(\ker T) - \dim(Y/T(X)).$$

If $\mathrm{ind}(T)$ is finite, T is called a Fredholm operator.

Properties of Fredholm operators:

- If $T\colon X \to Y$ is a Fredholm operator, then there exist a closed vector subspaces V of X and a finite dimensional subspace W of Y such that

$$X = \ker T \oplus V \quad \text{and} \quad Y = T(X) \oplus W.$$

  In particular, the surjective operator $T|_V\colon V \to T(X)$ is an isomorphism.
- If $T\colon X \to Y$ is a Fredholm operator, then $T^*\colon Y^* \to X^*$ is also a Fredholm operator and

$$\mathrm{ind}(T^*) + \mathrm{ind}(T) = 0.$$

- If $T\colon X \to Y$ and $S\colon Y \to Z$ are Fredholm operators, then $ST\colon X \to Z$ is also a Fredholm operator with

$$\operatorname{ind}(ST) = \operatorname{ind}(T) + \operatorname{ind}(S)\,.$$

- If $\dim X = \infty$ and $T\colon X \to Y$ is a Fredholm operator and $S\colon X \to Y$ is a strictly singular operator, then $T + S$ is a Fredholm operator with

$$\operatorname{ind}(T + S) = \operatorname{ind}(T)\,.$$

- If X is a Banach space and $S\colon X \to X$ is a strictly singular (in particular a compact) operator, then $I_X - \lambda S$ is a Fredholm operator for every λ and

$$\operatorname{ind}(I_X - \lambda S) = 0\,.$$

Theorem [Atkinson (1951)] For an operator $T\colon X \to Y$ between Banach spaces the following statements are equivalent:

(i) T is Fredholm operator;

(ii) There exist finite rank operators $K_1\colon X \to X$ and $K_2\colon Y \to Y$ and an operator $S\colon Y \to X$ such that

$$ST = I_X - K_1 \quad \text{and} \quad TS = I_Y - K_2\,.$$

Remarks (The Fredholm Alternative) If X is a Banach space and $K\colon X \to X$ is a compact operator, then for every $\lambda \neq 0$ exactly one of the following two exclusive statements is true:

(i) For every $y \in X$ the equation $x - \lambda Kx = y$ has a unique solution;

(ii) The equation $x - \lambda Kx = 0$ has a non-trivial solution.

When (ii) is true, the equation $x - \lambda Kx = 0$ has a finite number of linearly independent solutions.

Theorem Every Fredholm operator $T\colon X \to Y$ between Banach spaces has a pseudoinverse which is also Fredholm operator, i.e., such an operator $S\colon Y \to X$ satisfying:

$$TST = T.$$

In particular this yields that the equation $Tx = y$ has a solution if and only if Sy is a solution of this equation.

Introduction

- A mapping $F\colon \vec{\mathcal{B}} \to \mathcal{B}$ from the category $\vec{\mathcal{B}}$ of all couples of Banach spaces into the category $\mathcal{B}$ of all Banach spaces is said to be an interpolation functor if, for any couple $\vec{X} := (X_0, X_1)$, the Banach space $F(X_0, X_1)$ is intermediate with respect to $\vec{X}$ (i.e., $X_0 \cap X_1 \subset F(X_0, X_1) \subset X_0 + X_1$), and

$$T|_{F(X_0,X_1)}\colon F(X_0, X_1) \to F(Y_0, Y_1) \quad \text{for all } T\colon (X_0, X_1) \to (Y_0, Y_1);$$

- $T\colon (X_0, X_1) \to (Y_0, Y_1)$ means that $T\colon X_0 + X_1 \to Y_0 + Y_1$ is a linear operator such that the restrictions of T to the space X_j is a bounded operator from X_j to Y_j, for both $j = 0$ and $j = 1$.

- An interpolation method F is said to be regular if for every Banach couple (X_0, X_1) the intersection $X_0 \cap X_1$ is dense in $F(X_0, X_1)$.

- The real method For $\theta \in (0,1)$ and $q \in [1,\infty]$, $(X_0, X_1)_{\theta,q}$ is defined as the Banach space of all $x \in X_0 + X_1$ equipped with the norm

$$\|x\|_{\theta,q} = \left(\int_0^\infty \left[t^{-\theta} K(t, x; \vec{X}) \right]^q \frac{dt}{t} \right)^{1/q},$$

where

$$K(t, x; \vec{X}) := \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1};\ x = x_0 + x_1\}, \quad t > 0.$$

- For all $\theta \in (0,1)$ and $q \in [1,\infty)$, $F := (\,\cdot\,)_{\theta,q}$ is a regular interpolation functor.

- The complex method Let $S := \{z \in \mathbb{C};\, 0 < \mathrm{Re} z < 1\}$ be an open strip on the plane. For a given $\theta \in (0,1)$ and any couple $\vec{X} = (X_0, X_1)$ we denote by $\mathcal{F}(\vec{X})$ the Banach space of all bounded continuous functions $f\colon \bar{S} \to X_0 + X_1$ on the closure $\bar{S}$ that are analytic on S, and

$$\mathbb{R} \ni t \mapsto f(j+it) \in X_j, \quad j = 0, 1$$

is a bounded continuous function, and equipped with the norm

$$\|f\|_{\mathcal{F}(\vec{X})} = \max\left\{\sup_{t\in\mathbb{R}} \|f(it)\|_{X_0},\ \sup_{t\in\mathbb{R}} \|f(1+it)\|_{X_1}\right\}.$$

The Calderón (lower) complex interpolation space $[\vec{X}]_\theta := \{f(\theta);\, f \in \mathcal{F}(\vec{X})\}$ and is equipped with the norm:

$$\|x\|_\theta := \inf\left\{\|f\|_{\mathcal{F}(\vec{X})};\, x = f(\theta)\right\}.$$

Theorem

[*I. Ya. Shneiberg* (1974)] *If* $T\colon (X_0, X_1) \to (Y_0, Y_1)$ *is such that*

$$T\colon [X_0, X_1]_{\theta_*} \to [Y_0, Y_1]_{\theta_*}$$

is invertible (*resp., Fredholm*) *for some* $\theta_* \in (0,1)$, *then there exists* $\varepsilon > 0$ *such that*

$$T\colon [X_0, X_1]_{\theta} \to [Y_0, Y_1]_{\theta}$$

is invertible (*resp., Fredholm*) *and the index is constant for all* $\theta \in (\theta_* - \varepsilon, \theta_* + \varepsilon)$.

The majorization property for interpolation functors

An operator $T\colon (A_0, A_1) \to (B_0, B_1)$ between Banach couples is said to be invertible whenever the restriction $T|_{A_j}\colon A_j \to B_j$ is invertible (i.e., T is an isomorphism of A_j onto B_j) for both $j = 0$ and $j = 1$.

Lemma

Let (A_0, A_1) and (B_0, B_1) be Banach couples and let $T\colon (A_0, A_1) \to (B_0, B_1)$ be an invertible operator. Then, the following conditions are equivalent:

(i) $(T|_{A_0})^{-1}b = (T|_{A_1})^{-1}b, \qquad b \in B_0 \cap B_1\,;$

(ii) $T\colon A_0 + A_1 \to B_0 + B_1$ *is invertible*;

(iii) *For every interpolation functor F,*

$$T|_{F(A_0,A_1)}\colon F(A_0, A_1) \to F(B_0, B_1) \text{ is invertible.}$$

- Remark Let $\vec{X} = (X_0, X_1)$ be a complex couple and $T\colon (X_0, X_1) \to (X_0, X_1)$ be an operator. If $0 < \alpha < \beta < 1$ and $T_\alpha := T|_{[\vec{X}]_\alpha}$ and $T_\beta := T|_{[\vec{X}]_\beta}$ are invertible, then the inverses T_α^{-1} and T_β^{-1} do not coincide on $X_0 \cap X_1$ in general.

- Example The dilatation operator D_a ($a > 0$, $a \neq 1$) given by $D_a f(t) = f(at)$, $t > 0$ is bounded on $L^p = L^p(\mathbb{R}_+)$ for every $1 < p < \infty$ and

$$\sigma(D_a, L^p) = \left\{\lambda \in \mathbb{C};\, |\lambda| = a^{-1/p}\right\}.$$

If $|\lambda| = a^{-1/p}$, $p_0 < p < p_1$, then $T = \lambda I - D_a\colon (L^{p_0}, L^{p_1}) \to (L^{p_0}, L^{p_1})$ is invertible. Since

$$L^{p_0} = [L^1, L^\infty]_\alpha, \quad L^{p_1} = [L^1, L^\infty]_\beta$$

with $\alpha = 1 - 1/p_0$ and $\beta = 1 - 1/p_1$, it follows that T is not invertible on $L^p = [L^{p_0}, L^{p_1}]_\theta$, where $1/p = (1-\theta)/p_0 + \theta/p_1$.

Let F and G be interpolation functors. We have introduced the following key definitions:

- G is said to be majorized by F for invertibility if, for any Banach couples (X_0, X_1), (Y_0, Y_1) and any operator $T\colon (X_0, X_1) \to (Y_0, Y_1)$, invertibility of the operator $T|_{F(X_0,X_1)}\colon F(X_0, X_1) \to F(Y_0, Y_1)$ implies invertibility of

$$T|_{G(X_0,X_1)}\colon G(X_0, X_1) \to G(Y_0, Y_1)\,.$$

- G is said to be majorized by F for the Fredholmness property if, for any Banach couples (X_0, X_1), (Y_0, Y_1) and any operator $T\colon (X_0, X_1) \to (Y_0, Y_1)$ the Fredholmness of the operator $T|_{F(X_0,X_1)}\colon F(X_0, X_1) \to F(Y_0, Y_1)$ implies the Fredholmness of

$$T|_{G(X_0,X_1)}\colon G(X_0, X_1) \to G(Y_0, Y_1)\,.$$

Theorem

[*I. Asekritova, N. Kruglyak and M. M.*] *Suppose that the functor G is majorized by the regular functor F for invertibility. Then, for any regular Banach couples (X_0, X_1), (Y_0, Y_1) and any operator $T\colon (X_0, X_1) \to (Y_0, Y_1)$, the Fredholmness of the operator $T\colon F(X_0, X_1) \to F(Y_0, Y_1)$ implies the Fredholmness of the operator $T\colon G(X_0, X_1) \to G(Y_0, Y_1)$ with equality of indices*:

$$\mathrm{ind}(T|_{G(X_0,X_1)}) = \mathrm{ind}(T|_{F(X_0,X_1)}).$$

The above theorem in combination with our other results gives the following:

Theorem

The real interpolation functors

$$K_{\theta,q}(\,\cdot\,) := (\,\cdot\,)_{\theta,q}$$

are majorized by the functor $C_\theta(\,\cdot\,) := [\,\cdot\,]_\theta$ for the Fredholmness property.

Theorem

Let $T\colon (X_0, X_1) \to (Y_0, Y_1)$ *be an operator between couples of complex Banach spaces. If* $T\colon [X_0, X_1]_{\theta_*} \to [Y_0, Y_1]_{\theta_*}$ *is invertible for some* $\theta_* \in (0,1)$*, then*

$$T\colon (X_0, X_1)_{\theta_*,q} \to (Y_0, Y_1)_{\theta_*,q}$$

is invertible for all $q \in [1, \infty]$.

Theorem

If $T\colon (X_0, X_1) \to (Y_0, Y_1)$ *is such that* $T\colon [X_0, X_1]_{\theta_*} \to [Y_0, Y_1]_{\theta_*}$ *is Fredholm then for all* $1 \leqslant q \leqslant \infty$ *the operator*

$$T\colon (X_0, X_1)_{\theta_*,q} \to (Y_0, Y_1)_{\theta_*,q}$$

is Fredholm and index is the same

$$\mathrm{ind}(T|_{(X_0,X_1)_{\theta_*,q}}) = \mathrm{ind}(T|_{[X_0,X_1]_{\theta_*}}).$$

The uniqueness of inverses on intersection of interpolated Banach spaces

- A. P. Calderón (1983) If (Ω, Σ, μ) is a measure space and $T\colon L^p(\mu) \to L^p(\mu)$ is a bounded operator for $1 < p < \infty$, which is invertible for $p = 2$, then T is also invertible when $2 - \varepsilon < p < 2 + \varepsilon$, for some small $\varepsilon > 0$.

- A careful analysis of Calderón's proofs gives the compatibility of inverses, i.e., there exists some small $\varepsilon > 0$ such that for all $p, q \in (2 - \varepsilon, 2 + \varepsilon)$, the inverse T^{-1} considered on the space $L^p(\mu)$ is compatible with T^{-1} considered on $L^q(\mu)$ when both operators are restricted to $L^p(\mu) \cap L^q(\mu)$.

- J. Pipher and G. Verchota (1992) shown a useful application of Claderón's result for solvability of the Dirichlet problem in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with data in $L^p(\partial\Omega)$:

$$\Delta u = 0 \quad \text{in} \quad \Omega \quad \text{with} \quad u = f \quad \text{and} \quad \partial u/\partial n = g \text{ on } \partial\Omega\,.$$

Theorem

[*E. Albrecht and V. Müller* (2001)] *Let* (X_0, X_1) *be a complex Banach couple,* $T\colon (X_0, X_1) \to (X_0, X_1)$ *and let* $T_\theta := T|_{[X_0,X_1]_\theta}$ *for all* $\theta \in (0,1)$. *Suppose that, for some* $\alpha \in (0,1)$

$$T_\alpha \colon [X_0, X_1]_\alpha \to [X_0, X_1]_\alpha \quad \textit{is is invertible}.$$

Then there exists a neighbourhood $U \subset (0,1)$ *of* α *such that* T_θ *is invertible and the inverse operators* T_θ^{-1} *and* T_α^{-1} *agree on* $X_0 \cap X_1$ *for all* $\theta \in U$*, that is,*

$$T_\theta^{-1}(x) = T_\alpha^{-1}(x), \quad x \in X_0 \cap X_1 .$$

The result mentioned above was the main motivation for us to introduce the following:

Definition A family $\{F_\theta\}_{\theta\in(0,1)}$ of interpolation functors is said to be stable if for any Banach couples $\vec{A} = (A_0, A_1)$ and $\vec{B} = (B_0, B_1)$ and for every operator $S\colon \vec{A} \to \vec{B}$ such that the restriction S_{θ_*} of S to $F_{\theta_*}(\vec{A})$ is invertible for some $\theta_* \in (0, 1)$, there exists $\varepsilon > 0$ such that, for any $\theta \in I(\theta_*) = (\theta_* - \varepsilon, \theta_* + \varepsilon)$, we have

(i) $S_\theta\colon F_\theta(\vec{A}) \to F_\theta(\vec{B})$ is invertible operator;

(ii) $S_\theta^{-1}\colon F_\theta(\vec{B}) \to F_\theta(\vec{A})$ agrees with $S_{\theta_*}^{-1}\colon F_{\theta_*}(\vec{B}) \to F_{\theta_*}(\vec{A})$ on $B_0 \cap B_1$, i.e., $S_\theta^{-1}y = S_{\theta_*}^{-1}y$ for all $y \in B_0 \cap B_1$;

(iii) $\sup_{\theta\in I(\theta_*)} ||S_\theta^{-1}||_{F_\theta(\vec{B})\to F_\theta(\vec{A})} \leqslant C||S_{\theta_*}^{-1}||_{F_{\theta_*}(\vec{B})\to F_{\theta_*}(\vec{A})}$ for some $C = C(\theta_*)$.

Intervals of invertibility of interpolated operators

- Remark Let $\{F_\theta\}_{\theta\in(0,1)}$ be a stable family of interpolation functors and let $T\colon (X_0, X_1) \to (Y_0, Y_1)$. Then the set of all $\theta \in (0,1)$ for which

$$T\colon F_\theta(X_0, X_1) \to F_\theta(Y_0, Y_1)$$

is invertible, is open, so it is a union of open disjoint intervals. These intervals we will call intervals of invertibility of T with respect to the family $\{F_\theta\}_{\theta\in(0,1)}$.

- Question Let $I \subset (0,1)$ be any interval of invertibility of T. Is it true that for any θ, $\theta' \in I$ the inverses T_θ^{-1} and $T_{\theta'}^{-1}$ agree on

$$F_\theta(Y_0, Y_1) \cap F_{\theta'}(Y_0, Y_1)?$$

The above Question was the main motivation to give the following:

Definition A family of interpolation functors $\{F_\theta\}_{\theta\in(0,1)}$ satisfies the (Δ)-condition if for any Banach couple $\vec{A} = (A_0, A_1)$ and for any θ_0, θ_1
with $0 < \theta_0 < \theta_1 < 1$, we have continuous inclusions

$$F_{\theta_0}(\vec{A}) \cap F_{\theta_1}(\vec{A}) \hookrightarrow \Delta_{\theta_0<\theta<\theta_1}(F_\theta(\vec{A})) \hookrightarrow F_{\theta_0}(\vec{A})^c \cap F_{\theta_1}(\vec{A})^c,$$

where $\Delta_{\theta_0<\theta<\theta_1}(F_\theta(\vec{A}))$ is a Banach space equipped with the norm

$$\|a\|_{\Delta_{\theta_0<\theta<\theta_1}(F_\theta(\vec{A}))} := \sup_{\theta_0<\theta<\theta_1} \|a\|_{F_\theta(\vec{A})}\,.$$

and the Gagliardo completions $F_{\theta_0}(\vec{A})^c$ and $F_{\theta_1}(\vec{A})^c$ are both taken with respect to the interpolation sum $F_{\theta_0}(\vec{A}) + F_{\theta_1}(\vec{A})$.

Recall that a family of interpolation functors $\{F_\theta\}_{\theta\in(0,1)}$ satisfies the reiteration condition if for any Banach couple $\vec{A} = (A_0, A_1)$ and for any θ_0, θ_1, $\lambda \in (0,1)$, we have

$$F_\lambda(F_{\theta_0}(\vec{A}), F_{\theta_1}(\vec{A})) = F_{(1-\lambda)\theta_0+\lambda\theta_1}(\vec{A}).$$

Theorem

[I. Asekritova, N. Kruglyak and M. M.] Let $T\colon (X_0, X_1) \to (Y_0, Y_1)$ and let $I \subset (0,1)$ be an interval of invertibility of T with respect to the stable family of interpolation functors $\{F_\theta\}_{\theta\in(0,1)}$. Assume that $\{F_\theta\}_{\theta\in(0,1)}$ satisfy both the (Δ) and the reiteration condition. Then for any $\theta_0, \theta_1 \in I$, the inverse operators $T_{\theta_0}^{-1}$ and $T_{\theta_1}^{-1}$ agree on $F_{\theta_0}(\vec{Y}) \cap F_{\theta_1}(\vec{Y})$, i.e.,

$$T_{\theta_0}^{-1}(y) = T_{\theta_1}^{-1}(y), \quad y \in F_{\theta_0}(\vec{Y}) \cap F_{\theta_1}(\vec{Y}).$$

Application for classical interpolation methods

Based on general our results, we obtain the following applications:

Theorem

Let $1 \leqslant q \leqslant \infty$ *and let* $T\colon (X_0, X_1) \to (Y_0, Y_1)$ *and let* $I \subset (0,1)$ *be an interval of invertibility of* T *with respect to the family* $\{(\cdot)_{\theta,q}\}_{\theta\in(0,1)}$ *of real interpolation functors. Then for any* θ_0*,* $\theta_1 \in I$*,*

$$T_{\theta_0}^{-1}(y) = T_{\theta_1}^{-1}(y), \quad y \in (Y_0, Y_1)_{\theta_0,q} \cap (Y_0, Y_1)_{\theta_1,q}.$$

Theorem

[*N. Kalton, S. Mayboroda and M. Mitrea* (2007)] *Let* $T\colon (X_0, X_1) \to (Y_0, Y_1)$ *be an operator between couples of complex Banach spaces and let* $I \subset (0,1)$ *be an interval of invertibility of* T *with respect to the family* $\{[\,\cdot\,]_\theta\}_{\theta\in(0,1)}$*. Then for any* θ_0*,* $\theta_1 \in I$*,*

$$T_{\theta_0}^{-1}(y) = T_{\theta_1}^{-1}(y), \quad y \in [Y_0, Y_1]_{\theta_0} \cap [Y_0, Y_1]_{\theta_1}.$$

Let (X_0, X_1) be a Banach couple of complex Banach function lattices on a σ-finite measure space (Ω, Σ, μ). The Calderón product $X_0^{1-\theta} X_1^{\theta}$ $(0 < \theta < 1)$ is defined to be the space of all $f \in L^0(\mu)$ such that

$$|f| \leqslant \lambda\,|f_0|^{1-\theta}|f_1|^{\theta}, \quad \mu - a.e.$$

for some $\lambda > 0$ and $f_j \in X_j$ with $\|f_j\|_{X_j} \leqslant 1$, $j = 0, 1$. The Calderón product is a Banach function lattice on (Ω, Σ, μ) equipped with the norm

$$\|f\| := \inf\left\{\lambda > 0;\, |f| \leqslant \lambda\,|f_0|^{1-\theta}|f_1|^{\theta},\, f_0 \in B_{X_0},\, x_1 \in B_{X_1}\right\}.$$

Theorem

Let (X_0, X_1), $\vec{Y} = (Y_0, Y_1)$ be couples of Banach lattices with the Fatou property. Assume that $T\colon (X_0, X_1) \to (Y_0, Y_1)$ is such that $T\colon X_0^{1-\theta_*} X_1^{\theta_*} \to Y_0^{1-\theta_*} Y_1^{\theta_*}$ is an invertible operator for some $\theta_* \in (0, 1)$. Then there exists $\delta > 0$ such that

$$T\colon X_0^{1-\theta} X_1^{\theta} \to Y_0^{1-\theta} Y_1^{\theta}$$

is an invertible operator whenever $|\theta - \theta_*| < \delta$.

Stability of Fredholm properties on interpolation scales of Banach spaces

Definitions [following M. Cwikel, N. Kalton, M. Milman and R. Rochberg (2002)]

- Let **Ban** be the class of all Banach spaces over the complex field. A mapping $\mathcal{X}\colon \mathbf{Ban} \to \mathbf{Ban}$ is called a pseudolattice, or a **pseudo-$\mathbb{Z}$-lattice**, if
  (i) for every $B \in \mathbf{Ban}$ the space $\mathcal{X}(B)$ consists of B valued sequences $\{b_n\} = \{b_n\}_{n\in\mathbb{Z}}$ modelled on $\mathbb{Z}$;
  (ii) whenever A is a closed subspace of B it follows that $\mathcal{X}(A)$ is a closed subspace of $\mathcal{X}(B)$;
  (iii) there exists a positive constant $C = C(\mathcal{X})$ such that, for all A, $B \in \mathbf{Ban}$ and all bounded linear operators $T\colon A \to B$ and every sequence $\{a_n\} \in \mathcal{X}(A)$, the sequence $\{Ta_n\} \in \mathcal{X}(B)$ and satisfies the estimate

$$\|\{Ta_n\}\|_{\mathcal{X}(B)} \leqslant C\|T\|_{A\to B}\|\{a_n\}\|_{\mathcal{X}(A)};$$

  (iv)

$$\|b_m\|_B \leqslant \|\{b_n\}\|_{\mathcal{X}(B)}$$

  for each $m \in \mathbb{Z}$, all $\{b_n\} \in \mathcal{X}(B)$ and all Banach spaces B.

- For every Banach couple $\vec{B} = (B_0, B_1)$ and every Banach couple of pseudolattices $\vec{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$, let $\mathcal{J}(\vec{\mathcal{X}}, \vec{B})$ be the Banach space of all $B_0 \cap B_1$ valued sequences $\{b_n\}$ such that $\{e^{jn} b_n\} \in \mathcal{X}_j(B_j)$ $(j = 0, 1)$, equipped with the norm.

$$\|\{b_n\}\|_{\mathcal{J}(\vec{\mathcal{X}}, \vec{B})} = \max\left\{\|\{b_n\}\|_{\mathcal{X}_0(B_0)}, \|\{e^n b_n\}\|_{\mathcal{X}_1(B_1)}\right\}.$$

- For every s in the annulus $\mathbb{A} := \{z \in \mathbb{C};\, 1 < |z| < e\}$, we define the Banach space $\vec{B}_{\vec{\mathcal{X}},s}$ to consist of all elements of the form $b = \sum_{n\in\mathbb{Z}} s^n b_n$ (convergence in $B_0 + B_1$ with $\{b_n\} \in \mathcal{J}(\vec{\mathcal{X}}, \vec{B})$, equipped with the norm

$$\|b\|_{\vec{B}_{\vec{\mathcal{X}},s}} = \inf\left\{\|\{b_n\}\|_{\mathcal{J}(\vec{\mathcal{X}}, \vec{B})};\ b = \sum_{n\in\mathbb{Z}} s^n b_n\right\}.$$

It is easy to check that the map $\vec{B} \mapsto \vec{B}_{\vec{\mathcal{X}},s}$ is an interpolation functor.

- A couple $\vec{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ of Banach pseudolattices, is said to be translation invariant if for any Banach space B,

$$\big\|\{S^k(\{b_n\}_{n\in\mathbb{Z}}\}\big\|_{\mathcal{X}_j(B)} = \big\|\{b_n\}_{n\in\mathbb{Z}}\big\|_{\mathcal{X}_j(B)}$$

for all $\{b_n\}_{n\in\mathbb{Z}} \in \mathcal{X}_j(B)$, each $k \in \mathbb{Z}$ and $j \in \{0,1\}$, where is the left-shift operator defined by $S\{b_n\} = \{b_{n+1}\}$.
- $\vec{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ is said to be a rotation invariant Banach couple of pseudolattices whenever the rotation map

$$\{b_n\}_{n\in\mathbb{Z}} \mapsto \{e^{in\tau} b_n\}_{n\in\mathbb{Z}}$$

is an isometry of $\mathcal{X}_j(B)$ onto itself for every real τ and every Banach space B.

Theorem

[I. Asekritova, N. Kruglyak and M. M.] Let $\vec{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ *be a couple of rotation and translation invariant pseudolattices and let* $\{F_\theta\}_{\theta\in(0,1)}$ *be a family of interpolation functors given by* $F_\theta(X_0, X_1) := (X_0, X_1)_{\vec{\mathcal{X}}, e^\theta}$ *for any Banach couple* (X_0, X_1)*. Suppose that* F_θ *is regular functor and* $F_\theta(X_0, X_1) = F_\theta(X_0^\circ, X_1^\circ)$ *for any Banach couple* (X_0, X_1)*. If* $T\colon (X_0, X_1) \to (Y_0, Y_1)$ *is such that the operator*

$$T|_{F_{\theta_*}(X_0,X_1)}\colon F_{\theta_*}(X_0, X_1) \to F_{\theta_*}(Y_0, Y_1) \quad \text{is Fredholm.}$$

Then there exists $\varepsilon = \varepsilon(\theta_*, \vec{\mathcal{X}}) > 0$ *such that for any* $\theta \in (\theta_* - \varepsilon, \theta_* + \varepsilon)$ *the operator*

$$T|_{F_\theta(X_0,X_1)}\colon F_\theta(X_0, X_1) \to F_\theta(Y_0, Y_1)$$

is also Fredholm and $\operatorname{ind}(T|_{F_\theta(X_0,X_1)}) = \operatorname{ind}(T|_{F_{\theta_*}(X_0,X_1)})$*.*

Theorem

[I. Asekritova, N. Kruglyak and M. M.] Let $\vec{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ *be a Banach couple of translation invariant pseudolattices and let* $T\colon \vec{X} \to \vec{Y}$ *be an operator between complex Banach couples. Assume that* $T\colon \vec{X}_{\vec{\mathcal{X}},s} \to \vec{Y}_{\vec{\mathcal{X}},s}$ *is invertible for some* $s \in \mathbb{A}$*. Then* $T_\omega\colon \vec{X}_{\vec{\mathcal{X}},\omega} \to \vec{Y}_{\vec{\mathcal{X}},\omega}$ *is invertible for all* ω *in an open neighbourhood* $W = \{\omega \in \mathbb{A};\, |\omega - s| < r\}$ *of* s *in* $\mathbb{A}$ *with*

$$r = \big[2\delta(s)\big(1 + \|T\|_{\vec{X}\to\vec{Y}}\|T^{-1}\|_{\vec{Y}_{\vec{\mathcal{X}},s}\to\vec{X}_{\vec{\mathcal{X}},s}}\big)\big]^{-1},$$

where $\delta(s) = \max\big\{(|s|-1)^{-1},\, (e-|s|)^{-1}\big\}$*. Moreover the following upper estimate for the norm of* T_ω *holds,*

$$\big\|T_\omega^{-1}\big\|_{\vec{Y}_{\vec{\mathcal{X}},\omega}\to\vec{X}_{\vec{\mathcal{X}},\omega}} \leqslant 2\big\|T_s^{-1}\big\|_{\vec{Y}_{\vec{\mathcal{X}},s}\to\vec{X}_{\vec{\mathcal{X}},s}}, \quad \omega \in W.$$

Theorem

[I. Asekritova, N. Kruglyak and M. M.] Let $\vec{\mathcal{X}} = (\mathcal{X}_0, \mathcal{X}_1)$ *be a couple of translation and rotation invariant pseudolattices and let* $T\colon \vec{X} \to \vec{Y}$*. Assume that* $T_{\theta_*}\colon \vec{X}_{\vec{\mathcal{X}},e^{\theta_*}} \to \vec{Y}_{\vec{\mathcal{X}},e^{\theta_*}}$ *is invertible for some* $\theta_* \in (0,1)$*. Then* $T_\theta\colon \vec{X}_{\vec{\mathcal{X}},e^{\theta}} \to \vec{Y}_{\vec{\mathcal{X}},e^{\theta}}$ *is invertible for all* θ *in an open neighbourhood* $I = \{\theta \in (0,1);\, |\theta - \theta_*| < \varepsilon\}$ *of* θ_* *with*

$$\varepsilon = \Big[2e\eta(\theta_*)\big(1 + \|T\|_{\vec{X}\to\vec{Y}}\|T^{-1}\|_{\vec{Y}_{\vec{\mathcal{X}},e^{\theta_*}}\to\vec{X}_{\vec{\mathcal{X}},e^{\theta_*}}}\big)\Big]^{-1},$$

where $\eta(\theta_*) = \max\big\{(e^{\theta_*} - 1)^{-1},\, (e - e^{\theta_*})^{-1}\big\}$*. Moreover* T_θ^{-1} *agrees with* $T_{\theta_*}^{-1}$ *on* $Y_0 \cap Y_1$ *and*

$$\|T_\theta^{-1}\|_{\vec{Y}_{\vec{\mathcal{X}},e^{\theta}}\to\vec{X}_{\vec{\mathcal{X}},e^{\theta}}} \leqslant 2\|T_{\theta_*}^{-1}\|_{\vec{Y}_{\vec{\mathcal{X}},e^{\theta_*}}\to\vec{X}_{\vec{\mathcal{X}},e^{\theta_*}}}, \quad \theta \in I.$$